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Parameter addition to a family of multivariate exponential and weibull distribution

Akintunde Oyetunde A
Department of Mathematics, Faculty of Science, Federal University Oye-Ekiti, Ekiti State, Nigeria

Abstract
Various methods of introducing additional parameter to a family of multivariate exponential and Weibull distributions are presented. One of them is used to give a new two-parameter extension of the multivariate exponential distribution which may appear to be easier to deal with than those such commonly used two-parameter family of multivariate life distributions as the Weibull, gamma and log-normal distributions. Another general method that allows additional new three-parameter to a family of multivariate Weibull distribution is also introduced and studied. All the families of distributions expanded by either or both of these methods have the property that the minimum of a geometric number of independent random variables with common distribution in the family has a distribution also in the family.

Keywords: geometric, multivariate, life distribution, parameter family

1. Introduction

Exponential and Weibull distributions play important roles in the analysis of survival or life time data as discussed by Ali, Mikhail and Haq [1]. These two distributions play such roles simply because of their constant hazard rates, convenient statistical theory, as well as their important property of lacking of memory. Cox and Oakas [5] stated that whenever the one-parameter family of univariate or bivariate exponential distribution is found to be insufficient, a number of wider families such as Gamma, Weibull and Gompartz-Makeham distributions are mostly used instead. Also, Cox and Oakas [5] discussed the usefulness and importance of these distributions in detail. Johnson *et al*. [10] explained these families of distributions in broader way. Genest and his fellow researchers [8] presented the usefulness and important properties of these distributions in details.

There are many methods that can be used to introduce new parameters in order to expand and simplify families of distributions for either adding flexibility or to construct either covariate or correlation models. This is as stated clearly in Marshall and Oikin [14]. Whenever a scale parameter is added to a family of distributions, it accelerates life model and by taking powers of the bivariate survival function introduces a parameter that give rises to the proportional hazard rate model. According to Weibull [17] and Feller [7], the family of Weibull distributions contains the exponential distributions and it is constructed by taking the powers of exponentially distributed random variables. Similarly, the family of gamma distributions contains the exponential distributions but it is constructed by taking powers of the Laplace transform of the exponentially distributed random vectors. Arnoid [2] as well as Marshall and Oikin [14] presented and studied the method of adding parameter to a family of univariate exponential distributions in order to expand and become more flexible whenever new parameter is introduced into it. Marshall and Oikin [13] also studied the properties of the new families of these families of these distributions formed by addition of the new parameter

In this research paper, an attempt has been made to prevent and discuss a general method of adding new parameter in the families of multivariate exponential and Weibull distributions, in particular, starting with a multivariate survival function $\bar{F}(x_1, x_2, \dots, x_n)$, the one-parameter family of multivariate survival function:

$$\bar{G}(x_1, x_2, \dots, x_n; \alpha) = \frac{\alpha \bar{F}(x_1, x_2, \dots, x_n)}{1 - \overline{\alpha \bar{F}}(x_1, x_2, \dots, x_n)} = \frac{\alpha \bar{F}(x_1, x_2, \dots, x_n)}{F(x_1, x_2, \dots, x_n) + \alpha \bar{F}(x_1, x_2, \dots, x_n)} \quad \begin{matrix} -\infty < x_1, x_2, \dots, x_n < \infty \\ 0 < \alpha < \infty \end{matrix} \tag{1.1}$$

with $\bar{\alpha} = 1 - \alpha$. As in univariate and bivariate distributions cases, it also worth noting that $\bar{G} = \bar{F}$ whenever $\alpha = 1$.

The particular case that $\bar{F}(x_1, x_2, \dots, x_n)$ is an exponential distribution gives a new two-parameter family distributions that may sometimes be used in place of usual multivariate Weibull and gamma families of distributions. It should be noted that all the methods used in introducing an additional parameter have a stability property. That is, if the method is applied twice, nothing new is obtained the second time. Therefore, a power of an exponential random vectors have a multivariate Weibull distribution, but the power of a Weibull random vectors is nothing but another Weibull random vectors. Similarly, it is equation (1.1) above, a multivariate survival form of the form $\bar{G}$ is introduced for $\bar{F}$, then the equation (1.1) gives nothing new.

Multivariate Density and Hazard Rate of the new family of distributions

As far as the multivariate function $\bar{F}$ has a multivariate density function, then the multivariate survival function $\bar{G}$ stated in equation (1.1) has easily computed the multivariate densities. In particular, whenever $\bar{F}$ has a multivariate density $f(x_1, x_2, \dots, x_n)$ and rate of hazard $r_{\bar{F}}$, then the multivariate density function $\bar{G}$ has the multivariate density function $g(x_1, x_2, \dots, x_n)$ which is given by:

$$\bar{G}(x_1, x_2, \dots, x_n) = \frac{\alpha f(x_1 x_2 \dots x_n)}{\{1-\bar{\alpha}\bar{F}(x_1 x_2 \dots x_n)\}^2} = \frac{\alpha f(x_1 x_2 \dots x_n)}{\{1-\bar{F}(x_1 x_2 \dots x_n)+\alpha\bar{F}(x_1 x_2 \dots x_n)\}^2} = \frac{\alpha f(x_1 x_2 \dots x_n)}{\{F(x_1 x_2 \dots x_n)+\alpha\bar{F}(x_1 x_2 \dots x_n)\}^2} \quad (2.1)$$

and the corresponding hazard rate is given by:

$$r(x_1, x_2, \dots, x_n;\ \alpha) = \frac{1}{\{1-\bar{\alpha}\bar{F}(x_1, x_2, \dots, x_n)\}} r_F(x_1, x_2, \dots, x_n;\ \alpha)$$

$$= \frac{1}{\{1-(1-\alpha)\bar{F}(x_1 x_2 \dots x_n)\}} r_F(x_1, x_2, \dots, x_n;\ \alpha) = \frac{1}{\{F(x_1 x_2 \dots x_n)+\alpha\bar{F}(x_1 x_2 \dots x_n)\}} r_F(x_1, x_2, \dots, x_n;\ \alpha) \quad (2.2)$$

Hence,

$$\lim_{x_1\to\infty}\lim_{x_2\to\infty}\dots\lim_{x_n\to\infty} r(x_1, x_2, \dots, x_n;\ \alpha) = \lim_{x_1\to\infty}\lim_{x_2\to\infty}\dots\lim_{x_n\to\infty} \frac{r_F(x_1, x_2, \dots, x_n)}{\alpha}$$

Similarly, as in bivariate case, it is also true that

$$\lim_{x_1\to\infty}\lim_{x_2\to\infty}\dots\lim_{x_n\to\infty} r(x_1, x_2, \dots, x_n;\ \alpha) = \lim_{x_1\to\infty}\lim_{x_2\to\infty}\dots\lim_{x_n\to\infty} r_F(x_1, x_2, \dots, x_n)$$

From the result obtained in equation (2.2) and what was stated by Genest, Ghoudi and Rivest [8], we can establish the following:

$$\frac{r_F(x_1 x_2 \dots x_n)}{\alpha} \le r(x_1, x_2, \dots, x_n;\ \alpha) \le r_F(x_1, x_2, \dots, x_n),\ -\infty < x_1, x_2, \dots, x_n < \infty, \alpha \ge 1 \quad (2.3)$$

$$r_F(x_1, x_2, \dots, x_n) \le r(x_1, x_2, \dots, x_n;\ \alpha) \le \frac{r_F(x_1 x_2 \dots x_n)}{\alpha},\ -\infty < x_1, x_2, \dots, x_n < \infty, \alpha \le 1 \quad (2.4)$$

Similarly,

$$\bar{F}(x_1, x_2, \dots, x_n) \le \bar{G}(x_1, x_2, \dots, x_n;\ \alpha) \le \bar{F}^{1/n}(x_1, x_2, \dots, x_n),\ -\infty < x_1, x_2, \dots, x_n < \infty, \alpha \ge 1 \quad (2.5)$$

$$\bar{F}^{1/n}(x_1, x_2, \dots, x_n) \le \bar{G}(x_1, x_2, \dots, x_n;\ \alpha) \le \bar{F}(x_1, x_2, \dots, x_n),\ -\infty < x_1, x_2, \dots, x_n < \infty, \alpha \le 1 \quad (2.6)$$

Using the same equation (2.2) above, we can establish that $\frac{r(x_1 x_2 \dots x_n;\ \alpha)}{r_F(x_1 x_2 \dots x_n)}$ is an increasing function in $x_i, i = 1,2,..n$ for $\alpha \ge 1$ and it is a decreasing function in $x_i, i = 1,2,..n$ for $0 < \alpha \le 1$.

When $F(0, 0, \dots, 0) = 0$, the corresponding hazard rate $r(0, 0, \dots, 0;\ \alpha)$ at the origin of the multivariate function behaves quite differently than it does for the Weibull or gamma distributions; for both these families, the distribution can be an exponential distribution, or $r(0, 0, \dots, 0) = 0$ or $r(0, 0, \dots, 0) = \infty$, so that $r(0, 0, \dots, 0)$ is discontinuous in the shape parameter. This is not the case with the multivariate family having hazard rates as stated in equation (2.2). Therefore, the multivariate family may be useful to make multivariate function $F(x_1, x_2, \dots, x_n)$ easier to understand. However, in spite of what are already stated in both equations (2.3) as well as (2.4) above, it needs not be that multivariable function $F(x_1, x_2, \dots, x_n)$ and its corresponding multivariate survival function $G(x_1, x_2, \dots, x_n)$ are at all similar to each other.

3. A new family of two-parameter multivariate Exponential distributions

Given the multivariate function $\bar{F}(x_1, x_2, \dots, x_n) = exp\{-\sum_{i=1}^{n}\beta x_i\}$, the two-parameter family of multivariate survival function:

$$\bar{G}(x_1, x_2, \dots, x_n;\ \alpha, \beta) = \frac{1}{(\alpha-1)+e^{\sum_{i=1}^{n}\beta x_i}}, (x_i > 0\ for\ all\ i, and\ \alpha > 0, \beta > 0) \quad (3.1)$$

can be derived from equation (1.1). the multivariate exponential distribution can be obtained as a special case of (3.1) when $\alpha = \beta = 1$. When $\alpha = \beta \ge 1$, this multivariate distribution is the conditional multivariate distribution, given $z > 0$, of a random variable z with the multivariate logistic survival function:

$$\Pr(Z > z) = \frac{\alpha}{\left\{1-(1-\alpha)e^{\sum_{i=1}^{n}\beta x_i}\right\}},\ for -\infty < z < \infty \quad (3.2)$$

Regarding equation (3.1) above as a special case of equations (2.1) and (2.2), it can be seen that the multivariate survival function

$G(x_1, x_2, \dots, x_n)$ has the multivariate density function g which can be defined as:

$$g(x_1, x_2, \dots, x_n;\ \alpha, \beta) = \frac{\alpha\beta e^{-\sum_{i=1}^{n}\beta x_i}}{\left\{1-(1-\alpha)e^{\sum_{i=1}^{n}\beta x_i}\right\}^2} = \frac{\alpha\beta e^{\sum_{i=1}^{n}\beta x_i}}{\left\{e^{\sum_{i=1}^{n}\beta x_i}-(1-\alpha)\right\}^2}, (x_i > 0, V_i;\ \ \alpha > 0, \beta > 0$$

and the corresponding hazard rate of this multivariate density function is as given:

$$G(x_1, x_2, \dots, x_n;\ \alpha, \lambda) = \frac{\lambda}{\left\{1-(1-\alpha)e^{\sum_{i=1}^{n}\lambda x_i}\right\}} = \frac{\lambda e^{\sum_{i=1}^{n}\lambda x_i}}{\left\{e^{\sum_{i=1}^{n}\lambda x_i}-(1-\alpha)\right\}}, (x_i > 0, V_i;\ \ \alpha > 0, \lambda > 0$$

At this point, it should be noted that $r(x_1, x_2, \dots, x_n;\ 1, \lambda) = \ \lambda$, that is $r(x_1, x_2, \dots, x_n;\ \alpha, \lambda)$ is decreasing function in $x_i, V_i;\ for\ 0 < \alpha \leq 1$. Similarly, $r(x_1, x_2, \dots, x_n;\ \alpha, \lambda)$ is an increasing function in $x_i, V_i;\ for\ \alpha \geq 1.$

Considering equations (2.3) and (2.4), it can be seen that

$$\frac{\beta}{\alpha} \leq r(x_1, x_2, \dots, x_n;\ \alpha, \beta) \leq \beta, \quad -\infty < x_1, x_2, \dots, x_n < \infty, \alpha \geq 1 \tag{3.3}$$

$$\beta \leq r(x_1, x_2, \dots, x_n;\ \alpha, \beta) \leq \frac{\beta}{\alpha}, \quad -\infty < x_1, x_2, \dots, x_n < \infty, 0 \leq \alpha \leq 1 \tag{3.4}$$

$$e^{-\sum_{i=1}^{n}\beta x_i} \leq \bar{G}(x_1, x_2, \dots, x_n;\ \alpha, \beta) \leq e^{-\sum_{i=1}^{n}\beta x_i/\alpha}, \quad -\infty < x_1, x_2, \dots, x_n < \infty, \alpha \geq 1 \tag{3.5}$$

$$e^{-\sum_{i=1}^{n}\beta x_i/\alpha} \leq \bar{G}(x_1, x_2, \dots, x_n;\ \alpha, \beta) \leq e^{-\sum_{i=1}^{n}\beta x_i}, \quad -\infty < x_1, x_2, \dots, x_n < \infty, 0 \leq \alpha \leq 1 \tag{3.6}$$

As in bivariate case, in multivariate also, it is true that distribution with an increasing hazard rate is new and better than used. Similarly, distribution with a decreasing hazard rate is new and worse than used. This fact was earlier presented by Barlow and Proschan [4]. From the above fact, it follows that when multivariate random variables $x_1, x_2, \dots, x_n$ have the multivariate distribution $G(x_1, x_2, \dots, x_n)$, the conditional multivariate survival function satisfies:

$$P(X_i > x_i + i \dots . X_n > x_n + > i / X_i > x_i, \dots X_n > x_n) = \begin{cases} \leq Pr(X_1 > 1, \dots X_n > 1\ ) & (\alpha \geq 1) \\ \geq Pr(X_1 > 1, \dots X_n > 1\ ) & (0 < \alpha \leq 1) \end{cases}$$

Proposition: The multivariate function $\log g(x_1, x_2, \dots, x_n;\ \alpha, \beta)$ is convex for $0 < \alpha \leq 1$ and concave for $\alpha \geq 1$.

The above result can be shown by differentiating the multivariate $\log g(x_1, x_2, \dots, x_n;\ \alpha, \beta)$ n-times with respect to all variables $x_1, x_2, \dots, x_n$. This means that for $\alpha \leq 1$, the multivariate function $G(x_1, x_2, \dots, x_n)$ is a decreasing function. On the other hand, for $\alpha \geq 1$, $g(x_1, x_2, \dots, x_n;\ \alpha, \beta)$ in unimodal, with the mode of each of the n-variables given as:

$$mod = \begin{cases} 0, & (\alpha \leq 2) \\ \beta^{-1}\log(\alpha - 1) & (\alpha > 2) \end{cases}$$

Considering equations (3.5) and (3.6), it can be shown that the multivariate function $G(x_1, x_2, \dots, x_n)$ has finite moments of all positive orders. By computing directly, it can be verified that if these n-variables have distribution function $G(x_1, x_2, \dots, x_n;\ \alpha, \beta)$, then each of the n-variables has first moment given as:

$$E(x_i) = \ \frac{\alpha \log \alpha}{\beta(1-\alpha)}, for\ i = 1,2, \dots, n \tag{3.7}$$

The above expectations are always positive quantities. In particular, for the marginal distribution of random variable X_i, we have:

$$E(X_i{}^r) = \ r\ \int_o^\infty \bar{G}\ (x_i; \alpha, \beta) x_i{}^{r-1} dx_i = \frac{r\alpha}{\beta^r}\int_0^1 \left\{\frac{(-logp)^{r-1}}{1-(1-\alpha)p}\right\} dp \tag{3.8}$$

which when r=1 is substituted in it, gives equation (3.7). Similarly, for the marginal distribution of random variable x_i, the i[th] moment is also given as in equation (3.8) above X_2 replacing $X_1. the\ same\ argument\ is\ applied\ to\ all\ remaining\ n-2\ variables$. The Laplace transform of marginal distribution

g of each of the n-random variables $x_1, x_2, \dots, x_n$ can also be obtained as follows.

For random variable X_1, it is given as:

$$E\left[e^{-\beta X_1}\right] = \int_0^1 \left\{\frac{\alpha p^r}{(1-(1-\alpha)p)^r}\right\} dp \tag{3.9}$$

Similarly, that of random variable X_2 can be obtained in the same way as above by replacing X_1 with X_2. The same pattern is applied to all remaining-2 random variables.

Equations (3.8) and (3.9) can be expressed as infinite series as far as $|1-\alpha| \leq 1$. Based on this, the integrands of (3.8) and (3.9) can be expanded in a power series and the result be integrated term by term to generate the following for the random variable X_1:

$$E({X_1}^r) = \frac{\alpha}{\beta}\int_1^\infty {X_1}^{r-1} e^{-X_1} \sum_{j=1}^\infty \alpha^{-1} e^{-\beta j} dx_1 = \frac{\alpha}{\beta}\sum_{j=1}^\infty \frac{\alpha^{-j} r(r)}{(j+1)^r} (|1-\alpha| \leq 1)$$

and also

$$E\left(e^{-ix_1}\right) = \alpha \int_0^1 p^r \sum_{j=0}^\infty (j+1) p^j \alpha^{-1} dp = \alpha \sum_{j=0}^\infty \alpha^{-1} \frac{j+1}{i+j+1} \quad (|1-\alpha| \leq 1) \tag{3.10}$$

Similarly, that of marginal distribution of random variable X_2 can also be obtained in the same way by using the corresponding moments and Laplace transform of the random variable X_2. All others follow in the same way.

As a consequence of the above proposition as well as what was earlier presented by Karlin, S.; Proschan, F. and Barlow, R. E. [11], the total positivities properties yield moment inequalities that are not generally true. In particular, the coefficient of variation $\frac{\delta}{\mu}$ is less than 1 for $\alpha < 1$ and is greater than 1 when $\alpha < 1$, δ^2 is the variance and μ is the first moment of random variables $x_1, x_2, \dots, x_n$. It is also clear that the kth quartile $\overline{x_k}$ of $\bar{G}$ can be obtained by the relation: $\overline{x_k} = \frac{1}{\mu}\log(\frac{(1-k)+\alpha k}{1-k})$

Also, the median of each of the random variables $x_1, x_2, \dots, x_n$ is given by the formula:
Median of $X_i = \frac{\log(1+\alpha)}{\beta}, \forall i, i = 1,2,\dots,n$

From the above relations, it can be observed that median, mode and expectations of random variables $x_1, x_2, \dots, x_n$ are all increasing functions in α and decreasing function in the scale parameter β.
Considering the monotonic nature of log $x_i \, \forall i, i = 1,2,\dots,n$ and the values of random variables $x_1, x_2, \dots, x_n$ are all positive, it can be shown that: $mode\,(X_i) < med(X_i) < \frac{\alpha}{\beta} < E(X_i), \forall i, i = 1,2,\dots,n$. It should also be noted that $\lim_{\alpha\to\infty} \frac{\mathrm{mode}(X_1)}{E(X_1)} = \lim_{\alpha\to\infty} \frac{\mathrm{mode}(X_2)}{E(X_2)} = \cdots = \lim_{\alpha\to\infty} \frac{\mathrm{mode}(X_n)}{E(X_n)} = 1$. If all $E(X_i), i = 1,2,\dots,n$ are fixed constants, say equal 1, then the weak limit of $\bar{G}$, as α tends to infinity, is degenerate at point 1, while the limit is degenerate at point zero when α tends to zero. It also worth noting that $\lim_{X_1\to\infty} \lim_{X_2\to\infty} \dots \lim_{X_n\to\infty} r(x_1, x_2, \dots, x_n; \alpha, \beta) = \beta$ is bounded and continuous in the parameters just like gamma distribution and unlike Weibull distribution.

4. Extended multivariate Weibull distributions
Consider the multivariate Weibull survival function:

$$\bar{F}(x_1, x_2, \dots, x_n) = e^{-\sum_{i=1}^n (\beta x_i)^\lambda}, \; x_i \geq 0, \forall i, \; \lambda > 0, \beta > o \tag{4.1}$$

Then using equations (1.1) and (4.1) above, we can get the new three-parameter survival function

$$\bar{G}(x_1, x_2, \dots, x_n; \alpha, \beta, \lambda) = \frac{\alpha e^{-\sum_{i=1}^n (\beta x_i)^\lambda}}{1-(1-\alpha) e^{-\sum_{i=1}^n (\beta x_i)^\lambda}} \tag{4.2}$$

This geometric extreme stable extension of the multivariate Weibull distribution may sometimes be a competitor to the more usual three-parameter Weibull distribution with survival function:

$$\bar{F}(x_1, x_2, \dots, x_n;\ \alpha, \beta, \delta) = e^{-\beta \sum_{i=1}^{n}(x_i-\delta)^{\lambda}} \quad x_i \geq \delta, \forall i, \alpha > 0, \beta > 0, \lambda > 0, -\infty < \delta < \infty$$

If $X_i, i = 1,2, \dots, n$ have a multivariate exponential distribution with parameter β=1, then $\frac{x_1^{\frac{1}{\lambda}}}{\beta}, \frac{x_2^{\frac{1}{\lambda}}}{\beta}, \dots, \frac{x_n^{\frac{1}{\lambda}}}{\beta}$ have the survival function as given in equation (4.1) above. Similarly, if $X_i, i = 1,2, \dots, n$ have the survival function (3.1) with parameter β=1, then $\frac{x_1^{\frac{1}{\lambda}}}{\beta}, \frac{x_2^{\frac{1}{\lambda}}}{\beta}, \dots, \frac{x_n^{\frac{1}{\lambda}}}{\beta}$ have the survival function as stated in (4.2). Therefore, moments of survival function given in (4.2) can be obtained from non-integer moment function of equation (3.1). Hence, from equation (3.6), it can be seen that whenever the random variables $X_i, i = 1,2, \dots, n$ have a multivariate survival function as in equation (4.2), then:

$$E\left[\prod_{i=1}^{n} X_i^s\right] = \prod_{i=1}^{n} \frac{x_i^{\alpha}}{\lambda} \sum_{j=1}^{k} \frac{(1-\alpha)^{\lambda}}{(j+1)^{\lambda}} \Gamma\left(\frac{s}{\lambda}\right) \qquad |1-\alpha| \leq 1 \tag{4.3}$$

If $|1-\alpha| > 1$, then the moments can be obtained from the equation (3.4) by applying change of variables technique that was earlier applied in deriving equation (4.3). However, those moments cannot be stated in closed form; therefore, even the first moments of equation (4.2) must be obtained numerically. By expressing the moments as:

$$E\left[\prod_{i=1}^{n} X_i^s\right] = \int_0^{\infty}\int_0^{\infty} \dots \int_0^{\infty} x \prod_{i=1}^{n} x_i^{s-1}\, \bar{F}(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \quad s>0$$

It can be shown that:

$$\lim_{\lambda \to \infty} E\left[\prod_{i=1}^{n} X_i^s\right] = \beta^s, \quad s > 0$$

Of course, these are random variables that are degenerate at point $\frac{1}{\beta}$.

It should be noted that the density and hazard rate of the distribution given by the equation (4.2), can b obtained from equations (2.1) and (2.2). The hazard rate, particularly, is given by:

$$r(x_1, x_2, \dots, x_n;\ \alpha, \beta, \lambda) = \frac{\beta\lambda \prod_{i=1}^{n}(\beta x_i)^{\lambda-1}}{\left[1-(1-\alpha)e^{-\sum_{i=1}^{n}(\beta x_i)^{\lambda}}\right]}$$

In this function, it can be verified, by applying calculus, that its hazard rate is increasing if $\alpha \geq 1,\ \lambda \geq 1$ and decreasing if $\alpha \leq 1,\ \lambda \leq 1$. If $\lambda > 1$, then the hazard rate is initially increasing and eventually increasing, but there may be one interval where it is decreasing. On the other hand, if $\lambda < 1$, then the hazard rate is initially decreasing and eventually decreasing, but there may be one interval where it is increasing. The slope changes at those intervals are subtle and hence graphical method can be applied in this case.

5. Geometric-extreme stability of multivariate distribution

Let $\overline{X_\iota} - X_1^{(1)}, X_1^{(2)}, \dots;\ \overline{X_2} - X_2^{(1)}, X_2^{(2)}, \dots; \dots,\ \overline{X_N} - X_N^{(1)}, X_N^{(2)}, \dots;$ be the sequences of independent identically distributed random multivariate random vectors with distributions as stated in the family (1.1), and if N has a geometric distribution on {1,2,3,...}, then minimum and maximum of all $X_i, X_2, \dots X_N$ also have distributions in the family. To see why this property may be of interest, recall that the extreme value distributions for extrema, and as such they are sometimes useful approximation. In practice, a random vector of interest may be the extreme of only a finite, possibly random number N of random number N of random vectors. When it has a geometric distribution, the random vector has a particularly important stability property, just like that of extreme value distributions.

Assume that N is independent of $X_i, X_2, \dots, X_N$ with a geometric (p) distribution, that is:

$$P(N = n) = (1-p)^{n-1}p, n = 1,2,3 \dots$$

$$U_1 = \min\left\{X_1^{(1)}, X_2^{(1)}, \dots X_N^{(1)}\right\}, U_2 = \min\left\{X_1^{(2)}, X_2^{(2)}, \dots X_N^{(2)}\right\}, \dots\ U_N = \min\{X_1^{(N)}, X_2^{(N)}, \dots X_N^{(N)}\} \text{ and let} \tag{5.1}$$

$$V_1 = \max\left\{X_1^{(1)}, X_2^{(1)}, \dots X_N^{(1)}\right\}, V_2 = \max\left\{X_1^{(2)}, X_2^{(2)}, \dots X_N^{(2)}\right\}, \dots V_N = \max\{X_1^{(N)}, X_2^{(N)}, \dots X_N^{(N)}\}$$

5.1 Definition: If $F \in \tau$ implies that the distribution of $U_i(V_{i)}, (i = 1,2,3, \dots n)$ are in τ, then τ Is said to be geometric minimum stable (geometric maximum stable). If τ is both geometric-minimum and geometric-maximum stable, then τ is said to be geometric-extreme stable.

The term 'geometric-maximum stable' was discussed by Marshal *et al*. [14] and Rachev *et al*. [15] to describe a related but more restricted concept. They apply the term not to families of distribution but to individual distribution; in their submission, a distribution is geometric-maximum stable if the location-scale parameter family generated by the distribution is geometric-maximum stable in our sense. The two ideas essentially coincide for families τ that are parameterized by location and scale. Most of the families considered in the paper are not of that form, a notable exception being the logistic distribution. For instance, the family of logistic distributions and multivariate survival function of the form:

$$F(x_1, x_2, \dots, x_n) = \frac{1}{1 + \theta e^{\beta x_1 + \beta x_2 + \dots + \beta x_n}}, \qquad -\infty < x_1, x_2, \dots, x_n < \infty; \quad \alpha, \beta > 0$$

is a geometric-extreme family, indeed distribution in this family are geometric-stable even in the sense of Rachev *et al*. [15]. The fact that this family is geometric-minimum stable was utilized by Arnold [3] to construct a stationary process with logistic distributions, with multivariate survival function of the form:

$$\begin{aligned}\bar{G}(x_1, x_2, \dots, x_n) &= P(U_1 > x_1, U_2 > x_2, \dots, U_n > x_n \\ &= \sum_{n=1}^{\infty} \bar{F}^n(x_1, x_2, \dots, x_n)(1-p)^{n-1} \\ &= \frac{p\bar{F}(x_1, x_2, \dots, x_n)}{1-(1-p)\bar{F}(x_1, x_2, \dots, x_n)}, \quad -\infty < x_1, x_2, \dots, x_n < \infty \end{aligned} \tag{5.2}$$

As an extension of univariate and bivariate parametric family of distributions given by Marshal *et al*. [14], the multivariate parametric family of distributions stated in equation (5.2), is also geometric-minimum stable.

Similarly, for random variables $V_i, (i = 1,2, \dots, n)$, also given in equation (5.1) by using arguments similar to those used above, we can see that:

$$\begin{aligned}G(x_1, x_2, \dots, x_n) &= P(V_1 \le x_1, V_2 \le x_2, \dots, V_n \le x_n \\ &= \sum_{n=1}^{\infty} \bar{F}^n(x_1, x_2, \dots, x_n)(1-p)^{n-1} \\ &= \frac{p\bar{F}(x_1, x_2, \dots, x_n)}{1-(1-p)\bar{F}(x_1, x_2, \dots, x_n)}, \quad -\infty < x_1, x_2, \dots, x_n < \infty \end{aligned} \tag{5.3}$$

According to Marshal *et al*. [14], the multivariate parametric family, given in equation (5.3) above is geometric-maximum stable.
The multivariate families defined in equations (5.2) and (5.3) above combine together to give single parametric family $\xi = \xi(F(x_1, x_2, \dots, x_n)) = \{G(x_1, x_2, \dots, x_n; \alpha), \alpha > 0\}$ where $\bar{G}(x_1, x_2, \dots, x_n)$ is given in equation (1.1), with condition that in equation (5.2), $0 < \alpha = p \le 1$ and in equation (5.3), with $\alpha - \frac{1}{p} \ge 1$. At this point, it can be seen that $\bar{G}(x_1, x_2, \dots, x_n) = \bar{F}(x_1, x_2, \dots, x_n)$, hence $F(x_1, x_2, \dots, x_n) \epsilon \xi$, furthermore, it also worth noting that $F(x_1, x_2, \dots, x_n) \epsilon \xi$ is stochastically increasing function in α.

Proposition: The parametric family $\mathcal{F}$ of distributions of the form (1.1) is geometric maximum stable.

Proof: To verify this proposition, it is enough to verify closure of $\mathcal{F}$ under a kind of composition, as follows. Suppose that $\bar{G}(x_1, x_2, \dots, x_n) = \frac{k\bar{G}(x_1, x_2, \dots, x_n; \alpha)}{\{1-(1-k)\bar{G}(x_1, x_2, \dots, x_n; \alpha)\}}$, where $\bar{G}(x_1, x_2, \dots, x_n; \alpha)$ is as stated in equation (5.3). Therefore, $\bar{H}(x_1, x_2, \dots, x_n) = \frac{k\alpha\bar{F}(x_1, x_2, \dots, x_n)}{\{1-(1-\alpha k)\bar{G}(x_1, x_2, \dots, x_n)\}}$

This shows that $H(x_1, x_2, \dots, x_n) \epsilon \mathcal{F}$, and consequently, $\mathcal{F}$ has both geometric-maximum and geometric-minimum stability.

The proof of the above proposition also shows that if F is replaced by any other distribution in $\mathcal{F}$. Then that distribution will also generate $\mathcal{F}$.

Below are some properties of geometric-extreme stable families that worth noting. The same properties also hold for geometric-minimum and geometric-maximum stable families.

a. If P_1 and P_2 are geometric-extreme stable families, then $P_1 \cup P_2$ *and* $P_1 \cap P_2$ are also geometric-extreme stable families; the empty set is vacuously such a family.
b. For every distribution F that determines a geometric-extreme stable family P(F), if $G \in P(F)$ *then* $P(G) = P(F)$. Therefore, the minimal geometric stable families form a partition of the set of all distributions into a set of equivalence classes. In this case, a minimal geometric-extreme stable family is a family which is non-empty and has no non-empty geometric-extreme stable sub-family.
c. If F and G differ only by a scale (location) parameter, then P(F) can be derived from P(F) by a common scale (location) parameter change.
d. Assume that $F \in P$, this means that $\bar{F}(0) > 0$, and also $\bar{F}$ is given by the formulae:

$$\bar{F}(x_1, x_2, \dots, x_n) = \begin{cases} 1 & x_i \le 0, V_i, i = 1,2, \dots, n \\ \dfrac{\bar{F}(x_1, x_2, \dots, x_n)}{F(0)} & x_i \ge 0, V_i, i = 1,2, \dots, n \end{cases}$$

If F is geometric-extreme stable, then $\{F_t, \ F \in T\}$ is also geometric-extreme stable.

(a) Let F be a family of distribution functions, and also suppose that:

$$P_{\theta,\delta} = \{G: G(x_1, x_2, \dots, x_n) = F^{\theta}(x_1 - \delta, x_2 - \delta, \dots, x_n - \delta) \ for\ some\ F \in P\}$$

If P is geometric-extreme stable, then $P_{\theta,\delta}$ is geometric stable for all $\theta > 0$ and all real δ

6. Application of Geometric Distribution in extreme stability property

The geometric-extreme stability property of $\mathcal{F} = \mathcal{F}(F)$ is indeed important, and it largely depends upon the fact that a geometric sum of independent identically distributed random variables has a geometric distribution. This partially explains why random-minimum stability cannot be expected if the geometric distribution is replaced by some other distribution on $\{1,2,\dots\}$. Therefore, if the above fact is repeated with the assumption that N-1 has a Poisson distribution, and then $\mathcal{F}$ would be replaced by a family that would not be Poisson-extreme stable.

If F is a distribution function and $\bar{G}(x_1, x_2, \dots, x_n; \theta) = \sum_{n=1}^{\infty} \bar{F}^n(x_1, x_2, \dots, x_n) t_n(\theta)$ has the stability property then the discrete distribution must satisfy the functional equation:

$$\sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} e^{-m} t_m(\theta) \right\}^n t_n(\alpha) = \sum_{n=1}^{\infty} z^n t_n(K), \qquad 0 \le z \le 1$$

The only solution to this equation is the geometric distribution when some regularity conditions are applied.

7. Conclusion

The general method of introducing one-parameter into a family of multivariate distribution is developed and presented. The extended exponential distribution provide a new method of adding two-parameter to a family of multivariate distribution which may sometimes compete with multivariate Weibull and gamma families of distributions. New method for derivation of three-parameter type of Weibull family of distribution is introduced and discussed. It is also presented in this paper that all the methods of adding parameter to different families of different distributions commonly possessed stability properties.

8. References

1. Ali HM, Mikkhail NN, Haq MS. A class of bivariate distributions including the bivariate logistic. Journal of Multivariate Analysis. 1979; 8:405-512.
2. Arnold BC. Multivariate exponential distributions based on hierarchial successive damage. Journal of Applied Probability. 1975; 42:142-147.
3. Arnold BC. A logistic process constructed using geometric minimization. Statistical Probability Leter. 1989; 7:253-259.

4. Barlow RE, Proschan F. Statistical theory and reliability and life testing. New York; Holl, USA, 1984.
5. Cox DR, Oakes D. Analysis of survival data. London; Chaman and Hall, 1984.
6. Downton F. Bivariate exponential distributions in reliability. Journal of Research Statistical Society, 1970; B32:403-417.
7. Feller W. An introduction to probability. Theory and Its Application, 3rd Edition. New York, Wiley, USA, 1968.
8. Generst C, Ghoudi K, Rivest IP. A semi-parametric estimation procedure of dependence parameters in multivariate families of distributions. Biometrika. 1995; 82:543-552
9. Hawkes AG. A bivariate exponential distribution with applications to reliability. Journal of Research Statistical Society. 1992; B34:129-131.
10. Johnson NI, Kotz S, Balakrishman N. Continuous univariate distributions I, Second Edition, New York, Wiley, USA, 1994.
11. Karlin S, Proschan F, Barlow RE. Moment inequalities of Polya frequency functions, Pacific Journal of Mathematics. 1961; 11:1023-1033.
12. Marshall AW, Oikin I. A family of bivariate distributions generated by the bivariate Bernoulli distribution. Journal of American Statistical Association. 1985; 80:332-338.
13. Marshal AW, Oikin I. Families of multivariate distributions. Journal of American Statistical Association. 1988; 83:834-841.
14. Marshal AW, Oikin I. A new method for adding a parameter to a family of distributions with applicationto the exponential and Weibull families. Biometrika. 1997; 84:641-652.
15. Rachev ST, Resnick S. Maximum geometric infinite divisibility and stability. Commun. Statist. Stoch. Mod. 1991; 7:191-218.
16. Umar SM. A new method of adding parameter to a family of bivariate exponential and Weibull distributions. Journal of the Nigerian Association of Mathematical Physics. 2009; 18:325-332.
17. Weibul W. A statistical distribution of wide applicability. Journal of Applied Mech. 1951; 18:293-297.